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The Nash modification and hyperplane sections on surfaces

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Abstract. We prove that the planar components of the tangent cone of a complex analytic surface at a point correspond to the base points of hyperplane sections by the Nash modification. This correspondence is then used to characterize domination relations between the normalized Nash modification and the normalized blow-up of a point.

Keywords: Nash modification, base points, tangent cone.

Mathematical subject classification: 32S05, 32S25, 32S45, 14B05.

1 Introduction

For studying singularities of germs at a point of complex analytic surfaces, two particular modifications may be considered: the blow-up of the point and the Nash modification. Both transformations have desingularization virtues. In fact, the surface can be desingularized after a finite iteration of normalized point blow-ups ([11], [1]) or normalized Nash modifications ([9]).

The domination relation between these two modifications is related to hyperplane sections and polar curves and their base points after one or another modification.

It is well known that the normalized blow-up of a point factors through the Nash modification if and only if the family of local (absolute) polar curves does not have a base point after the blow-up ([3], [9]). These base points correspond to the so called “exceptional tangents" of the surface at the blown-up point ([6]). They are completely characterized in the case of normal surfaces in [8]. For the case of hypersurfaces of $\mathbb{C}^3$ with non-isolated singularities we refer to [5].

In this work, we give a necessary and sufficient condition for the normalized Nash modification to factor through the blow-up of a point.

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We first characterize the base points of hyperplane sections after Nash modification. We prove that these base points are in one-to-one correspondence with the planar components of the tangent cone of the surface at the considered point (Theorem 3.2).

Then we prove that the normalized Nash modification of the surface factors through the blow-up of a point if and only if the tangent cone of the surface at that point does not have any planar component (Theorem 4.2).

In the last section, we use the characterizations of the base points of the polar curves after the blow-up of a point, given in [8], to prove that, in the case of normal surfaces, the normalized blow-up of a point dominates the normalized Nash modification if and only if they are isomorphic (Theorem 5.4).

The main tool we use to prove the results of this work is the so-called "normal-conormal" diagram, adapted to the case of the Nash modification (see for example [6]).

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2 Planes of the tangent cone

Let $(S, 0)$ be a germ of reduced and equidimensional complex analytic surface embedded in $(\mathbb{C}^N, 0)$.

Denote by $e : S' \to S$ the blow-up of the origin in a representative of $(S, 0)$; or equivalently the blow-up of the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{S,0}$ of holomorphic functions on $(S, 0)$. Call S_0 the non-singular locus of the surface S, and consider the morphism $\lambda : S_0 \to \mathbf{G}(2, N)$ such that $\lambda(x) = T_x S$; where $\mathbf{G}(2, N)$ is the Grassmannian of 2-dimensional linear subspaces of $\mathbb{C}^N$ and $T_x S$ is the direction of the tangent space to S at x. The closure $\tilde{S}$ of the graph of λ in $S \times \mathbf{G}(2, N)$ is a reduced analytic surface. The induced morphism $\nu : \tilde{S} \to S$ is called the Nash modification of S; it is an isomorphism over the non-singular locus of S (see for example [3], [5] or [9]). The blow-up of the ideal $\mathfrak{m}\mathcal{O}_{\tilde{S}}$, defines a morphism $e' : X \to \tilde{S}$.

We have the following commutative diagram:

$$\begin{array}{ccccccc} S \times \mathbb{P}^{N-1} \times \mathbf{G}(2, N) & \supset & X & \xrightarrow{e'} & \tilde{S} & \subset & S \times \mathbf{G}(2, N) \\ & & \downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \nu} & & \\ S \times \mathbb{P}^{N-1} & \supset & S' & \xrightarrow{e} & S & \subset & \mathbb{C}^N \end{array} \tag{1}$$

where the morphism ν' is induced by the universal property of the blowing-up. Let us call $\xi : X \to S$ the composed morphism $\nu \circ e' = e \circ \nu'$.

Recall the following description of the fibers of the morphisms e and ν:

$|e^{-1}(0)|$ is isomorphic to $|\mathrm{Proj}\, C_{S,0}|$, where $|\mathrm{Proj}\, C_{S,0}|$ is the set of generatrices of the tangent cone of S at the origin (see [10, §8]). In other words, a line l represented by a point of $\mathbb{P}^{N-1}$ is such that $(0, l) \in |e^{-1}(0)|$ if and only if there exists a sequence (x_n) of points in S converging to 0 such that the sequence of lines $(0x_n)$ (called secants) converges to l in $\mathbb{P}^{N-1}$.

On the other hand, $|\nu^{-1}(0)|$ is the set of all limits of directions of tangent spaces at the origin. In other words, $(0, T) \in |\nu^{-1}(0)|$ if and only if there exists a sequence (x_n) of non-singular points in S converging to 0 such that the sequence $(T_{x_n} S)$ converges to T in $\mathbf{G}(2, N)$.

Thanks to Whitney's lemma (see [10, theorem 22.1]), we can give the following description of the fibers of the morphism ξ (see also [5, proof of theorem 1.4.4.1]):

A point of $\xi^{-1}(0)$ is of the form $(0, l, T)$ where l and T are respectively limits of secants and directions of tangent spaces reached by the same sequence of non-singular points; in particular we have $l \subset T$.

Consider now the irreducible decomposition $|\xi^{-1}(0)| = \bigcup_\alpha D_\alpha$. All the components D_α are of dimension 1. Call $W_\alpha = |e'(D_\alpha)|$, and $V_\alpha = |\nu'(D_\alpha)|$. The duality between the V_α's and the W_α's is important for the description of the limits of tangent spaces (see [6] and [2]). We are going to use this duality to describe some particular components of the tangent cone of the surface at the origin; namely the planar components.

Proposition 2.1. *The sets W_α of dimension 0 are in one-to-one correspondence with the 2-dimensional planes of the tangent cone $|C_{S,0}|$.*

Proof. If $\dim W_\alpha = 0$ then it is of the form $(0, T_\alpha)$ with $T_\alpha \in \mathbf{G}(2, N)$. The component D_α being of dimension one, there exists an irreducible component C_α of $|C_{S,0}|$ such $D_\alpha = \{(0, l, T_\alpha), l \in Proj\ (C_\alpha)\}$. By the description we gave for $\xi^{-1}(0)$, we have $C_\alpha \subset T_\alpha$. Since T_α is a two-dimensional plane, we have $C_\alpha = T_\alpha$.

If T_0 is a two-dimensional plane contained in the reduced tangent cone of S at 0, then T_0 is a limit of tangent spaces to S at 0 ([4, theorem 1.5]). Hence $D_{\alpha_0} = \{(0, l, T_0), l \in \mathrm{Proj}\ (T_0)\}$ is an irreducible component of $|\xi^{-1}(0)|$. The image $W_{\alpha_0} = e'(D_{\alpha_0}) = \{(0, T_0)\}$ is of dimension 0. □

Remark 2.2. *Note that a W_α of dimension 1 is an irreducible component of the fiber $|\nu^{-1}(0)|$, in case it has dimension one. By the characterization given* *in [6], it is either a pencil of planes containing an exceptional tangent or the planes tangent to a non-planar irreducible component of the tangent cone at the origin.*

3 Base points of hyperplane sections by the Nash modification

D.T. Lê and B. Teissier showed that the exceptional tangents correspond exactly to the base points of the local (absolute) polar curves after the blow-up of the maximal ideal ([6, proposition 2.2.1]).

We are going to prove a dual statement, making a correspondence between the planes of the tangent cone and the base points of hyperplane sections after the Nash modification.

Definition 3.1. *Let $(C_\alpha, 0)$ be a family of germs of curves on the surface $(S, 0)$, parameterized by a projective space $\mathbb{P}^r$. Consider a modification of the surface $\mu : X \to S$.*

A point $\eta \in X$ is a base point of the family of curves (C_α) by the modification μ if there exists an open dense set $\Omega \subset \mathbb{P}^r$, such that η is a point of the strict transform of the curve C_α for any $\alpha \in \Omega$.

Theorem 3.2. *A point $\eta \in \tilde{S}$ is a base point of the hyperplane sections of S by the Nash modification if and only if η corresponds to a plane of the tangent cone of S at 0.*

Proof. Set $\eta = (0, T_0) \in \nu^{-1}(0)$, and suppose T_0 is a plane of the tangent cone $C_{S,0}$.

By Proposition (2.1), there exists α such that $\{(0, T_0)\} = W_\alpha = e'(D_\alpha)$. On the other hand, $V_\alpha = \nu'(D_\alpha)$ is an irreducible component of Proj $|C_{S,0}|$ that is actually Proj (T_0).

For any hyperplane $H \in (\mathbb{P}^{N-1})^\vee$, we have Proj $(H) \cap$ Proj $(T_0) \neq \emptyset$. Hence by commutativity of the diagram (1), the strict transform of $H \cap S$ by ν contains the point η; so η is a base point.

Conversely, suppose η is a base point of the hyperplane sections of S by ν. By commutativity of the diagram (1), for a generic hyperplane H, the intersection Proj $(H) \cap \nu'(e'^{-1}(\eta))$ is not empty. Hence, dim $e'^{-1}(\eta) > 0$. The point η corresponds then to a W_α of dimension zero, which is, by Proposition 2.1, a plane of the tangent cone. □

Remark 3.3. *From the proof of theorem 3.2, it follows that if the tangent cone $C_{S,0}$ contains a plane then, the strict transforms by the Nash modification of all* *the hyperplane sections (not only generical ones) will contain the point corresponding to that plane.*

4 Factorization of the Nash modification through the blow-up of the maximal ideal

It is known that the Nash modification has a “universal property" with respect to polar curves, in the sense that a normalized modification μ of the origin does not have any base point for the polar curves if and only if μ factorizes through the normalized Nash modification (see [9, III.1.2] and [3, 1.2]).

In this section we will state a similar result with respect to hyperplane sections and the blow-up of the maximal ideal.

We first prove an algebraic version of the statement that seems to be well known to many specialists.

Let $f_1, \ldots, f_r$ be holomorphic functions in $\mathcal{O}_{S,0}$, whose unique common zero in a sufficiently small neighborhood of the origin is 0. The linear system of curves generated by the f_i's is the family of curves defined by an equation of the form $\alpha_1 f_1 + \ldots + \alpha_r f_r = 0$ with $(\alpha_1 : \ldots : \alpha_r) \in \mathbb{P}^{r-1}$.

Proposition 4.1. *Let $\mu : X \to S$ be a normalized modification of the surface S. The linear system of curves generated by $f_1, \ldots, f_r$ has no base point by μ if and only if the ideal sheaf $(f_1, \ldots, f_r)\mathcal{O}_X$ is locally principal.*

Proof. Let us call I the ideal of $\mathcal{O}_{S,0}$ generated by $f_1, \ldots, f_r$.

Suppose $\eta \in \mu^{-1}(0) \subset X$ is not a base point by μ of the linear system generated by $f_1, \ldots, f_r$. Then there exists a linear combination f of the f_i's such that:

i) The strict transform of $f = 0$ by μ does not contain η, and

ii) The valuation of f along the irreducible components of the exceptional divisor passing through η is minimum among the valuations of all the functions in I.

Let $g \in I$. Since X is a normal surface, the exceptional divisor of μ is Cartier outside a finite number of points. So the quotient $(g \circ \mu)/(f \circ \mu)$ is well defined near η except maybe in η. Again by normality of X, this quotient extends to a holomorphic function near η. Hence $(f \circ \mu)$ generates $I\mathcal{O}_{X,\eta}$.

Conversely, suppose $I\mathcal{O}_{X,\eta}$ principal, with $\eta \in \mu^{-1}(0) \subset X$. If f_{i_0} is such that the order at η of $(f_{i_0} \circ \mu)$ is minimum among the orders at η of the other generators, then $I\mathcal{O}_{X,\eta} = (f_{i_0} \circ \mu)\mathcal{O}_{X,\eta}$. Since the ideal I is primary for the maximal ideal of $\mathcal{O}_{S,0}$, the strict transform of $f_{i_0} = 0$ by μ does not contain η. So, for a generic linear combination $f = \alpha_1 f_1 + \ldots + \alpha_r f_r$ the strict transform by μ of $f = 0$ does not contain the point η. Hence η is not a base point. □

If we apply Proposition 4.1 to the Nash modification, consider the case $I = \mathfrak{m}$ and use the universal property of the blowing-up and theorem 3.2, then we obtain:

Theorem 4.2. *The normalized Nash modification of a reduced equidimensional germ of surface* $(S, 0)$ *dominates the blow-up of the origin if and only if the reduced tangent cone of S at 0 does not contain any two-dimensional plane.*

5 Comparison between Nash modification and point blow-up for normal surfaces

We already know that, the normalized blow-up of the origin of a normal surface dominates the Nash modification if and only if there are no base points of the polar curves by the blow-up of the origin ([9, theorem III, 1.2]).

In [8, theorem 5.8], we gave characterizations of base points of polar curves on a normal surface by the blow-up of the origin. We used for that the fact that they correspond one-to-one to the exceptional tangents of the surface at 0 (see [6, proposition 2.2.1]).

Let us state this result for the commodity of the reader:

Theorem 5.1. *Let* $(S, 0)$ *be a germ of a normal surface singularity. Call* $e : S' \to S$ *the blow-up of the origin, and* $n : \bar{S} \to S'$ *its normalization. The base points of the family of polar curves by the blow-up e are:*

- *the image by n of the singular points of the surface* $\bar{S}$*,*
- *the image by n of the singular points of the exceptional divisor* $|(e \circ n)^{-1}(0)|$*,*
- *the critical values of the restriction of n to the exceptional divisor* $|(e \circ n)^{-1}(0)|$*, and*
- *the singular points of the analytically irreducible components of the exceptional divisor* $|e^{-1}(0)|$*.*

As a corollary of this theorem we have the following properties for the normal surfaces without base points for the polar curves by the blow-up of the origin.

Corollary 5.2. *Let* $(S, 0)$ *be a germ of a normal surface such that the family of polar curves does not have base points by the blow-up of the origin. Then, the normalized blow-up of the origin is smooth and the tangent cone at the origin is irreducible.*

Proof. The smoothness of the normalized blow-up is immediate from Theorem 5.1.

If the reduced tangent cone is not irreducible, then the exceptional divisor $|(e \circ n)^{-1}(0)|$ will have at least two irreducible components. By Zariski's main theorem, there is at least one singular point of the exceptional divisor $|(e \circ n)^{-1}(0)|$. By theorem 5.1, the image of such a point will be a base point of the polar curves by e. So the reduced tangent cone needs to be irreducible. □

Remark 5.3. *Notice that, in theorem 5.1, an intersection point of two irreducible components of* $e^{-1}(0)$ *does not need to be a base point of the polar curves by e (unless it is one of the other points specified in theorem 5.1). However, in the case of hypersurfaces of* $\mathbb{C}^3$*, or normal singularities whose blow-up at the origin is still normal, these intersection points are always base points of the polar curves (see [4, theorem 3.1] and [8, corollary 5.11] respectively).*

We can now state and prove the main result of this section:

Theorem 5.4. *Let* $(S, 0)$ *be a singular germ of a normal surface. The normalized blow-up of the origin dominates the normalized Nash modification if and only they are isomorphic. In this case they both desingularize the surface.*

Proof. Suppose that the normalized Nash modification does not dominate the blow-up of the origin. By theorem 4.2, the tangent cone $C_{S,0}$ contains a two-dimensional plane. Two cases are possible:

i) The reduced tangent cone is not irreducible. By corollary 5.2, the polar curves have a base point by the blow-up of the origin. Hence, by [9, theorem III, 1.2], the normalized blow-up of the origin does not dominate the Nash modification.

ii) $C_{S,0}$ is irreducible and is a two-dimensional plane. In this case, consider a linear space $L \subset \mathbb{C}^N$ of dimension $N - 2$, such that $L \cap |C_{S,0}| = \{0\}$. Call $p : \mathbb{C}^N \to \mathbb{C}^2$ the linear projection whose kernel is L. The restriction of p to S is a finite generic map $\pi : S \to \mathbb{C}^2$. Since the surface S is singular at 0, the discriminant of π is a non-empty curve (see [7, proposition 2.3]). Let D be a line of the tangent cone of the discriminant at 0. By, [8, theorem 3.3], the hyperplane $H = p^{-1}(D)$ is a limit of tangent hyperplanes to S at 0. Because of the condition $L \cap |C_{S,0}| = \{0\}$, the hyperplane H can not contain the tangent plane to the tangent cone (that is the tangent cone itself). So H contains an exceptional tangent (see [6] or [8]). Hence the polar curves have a base point by the normalized blow-up of the origin. So, by [9, theorem III, 1.2], the normalized blow-up of the origin does not dominate the Nash modification.

We have then proved that, if the normalized blow-up of the origin dominates the normalized Nash modification then the converse is also true, and hence they are isomorphic.

The other implication is obviously true. The smoothness is given by Corollary 5.2. □

6 Examples

i) Consider the surface S defined in $\mathbb{C}^3$ by the equation $f = x^n + y^n + z^n = 0$. f being homogeneous, the tangent cone $C_{S,0}$ is also defined by $f = 0$. It is an irreducible cone that is not a plane. So the normalized Nash modification dominates the blow-up of the origin (theorem 4.2).

We can prove independently of the results contained in this work, that the normalized Nash modification of this surface is isomorphic the the blow-up of the origin. In fact, the Nash modification is the blow-up of the jacobian ideal $(x^{n-1}, y^{n-1}, z^{n-1})$. The jacobian ideal has the same integral closure as the ideal $(x, y, z)^{n-1}$, and this last one has the same blow-up as the maximal ideal (x, y, z). So the normalized blow-up of the jacobian ideal is isomorphic to the blow-up of the maximal ideal (that is already normal).

ii) S is defined in $\mathbb{C}^3$ by $x^2 + y^2 + z^3 = 0$. The tangent cone is a union of two planes. So the normalized Nash modification does not dominate the blow-up of the origin (theorem 4.2).

Actually, the blow-up of the origin is the minimal resolution of $(S, 0)$. On the other hand the normalized Nash modification produces two singular points of multiplicity 3 each one (see for example [3]).

iii) Consider the surface S union of two planes in $\mathbb{C}^4$ intersecting at the origin. The Nash modification of S is the normalization of S, that is non-singular. The blow-up of the origin is a resolution of the singularity; it factors then through the normalization. But they are not isomorphic since the blow-up is not finite. This proves that we can not extend theorem 5.4 to non-normal surface singularities.

References

[1] R. Bondil and D.T. Lê, *Résolution des singularités de surfaces par éclatements normalisés (multiplicité, multiplicité polaire, et singularités minimales)*, Trends in singularities, Trends in Math., Birkhäuser Verlag, (2002), 31–81.

[2] G. González-Sprinberg, *L'obstruction locale d'Euler et le théorème de MacPherson*, Astérisque **82-83** (1981), 7–32.

[3] G. González-Sprinberg, *Résolution de Nash des points doubles rationnels*, Ann. Inst. Fourier **32**(2) (1982), 111–178.

[4] D.T. Lê and J.P.G. Henry, *Limites d'espaces tangents*, Fonctions de plusieurs variables complexes II, (Séminaire Norguet, 1974-75), vol. 482, Springer Lecture Notes in Math., (1975), 251–265.

[5] D.T. Lê and B. Teissier, *Sur la géométrie des surfaces complexes I. Tangentes exceptionnelles*, Amer. J. of Math. **101** (1979), 420–452.

[6] D.T. Lê and B. Teissier, *Limites d'espaces tangents en géométrie analytique*, Comment. Math. Helvetici **63** (1988), 540–578.

[7] J. Snoussi, *Limites d'espaces tangents à une surface normale*, Ph.D. thesis, Université de Provence, France, 1998.

[8] J. Snoussi, *Limites d'espaces tangents à une surface normale*, Comment. Math. Helv. **76** (2001), 61–88.

[9] M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. **131** (1990), 411–491.

[10] H. Whitney, *Tangents to an analytic variety*, Ann. of Math. **81**(2) (1965), 496–549.

[11] O. Zariski, *The reduction of singularities on an algebraic surface*, Ann. of Math. **40** (1939), 639–689.

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